

Noncommutative deformations of quantum field theories, locality and causality

M.A. Soloviev

J. Phys A, 40 (2007) 14593; Phys. Rev. D 77 (2008) 125013
arxiv: 0708.1151, 0802.0997 [hep-th]

Lebedev Institute, Moscow

SFT09, April 17, 2009

Motivation and Program

Restrictions imposed by the uncertainty principle and gravity on measurements

Low-energy limit of string theory



$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$$



Star products: Moyal-Weyl-Grönewold, Wick-Voros



(Galluccio, Lizzi, Vitale, (2008))



(Balachandran, Martone, (2009))

Twisted Poincaré covariance and twisted tensor product \otimes_\star



Under what conditions do the star and twisted tensor products converge?

How should the causality principle be implemented in NC QFT?

Plan

- Introduction: Star products and twisted tensor products
- Noncommutative deformations of quantum field theories
- Two concepts of wedge-locality
- Convergence of star products and adequate function spaces
- θ -Locality instead of microcausality
- Conclusions

Moyal \star -product

$$\begin{aligned}(f \star_M g)(x) &= f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right) g(x) \\ &= f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} f(x) \partial_{\nu_1} \dots \partial_{\nu_n} g(x)\end{aligned}$$

Wick-Voros \star -product

$$\begin{aligned}(f \star_V g)(x) &= f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu + \frac{\theta}{2} \overleftarrow{\partial}_\mu \overrightarrow{\partial}^{\dot{\mu}}\right) g(x) \\ T(f \star_M g) &= T(f) \star_V T(g), \quad T = \exp\left(\frac{\theta}{4} \nabla^2\right) \quad (\text{Berezin, 1971})\end{aligned}$$

Schwartz space of smooth functions of fast decrease

$$S(\mathbb{R}^d) = \left\{ f: \sup_x (1 + |x|)^N |\partial^\kappa f(x)| < \infty, N \in \mathbb{Z}_+, \kappa \in \mathbb{Z}_+^d \right\}, \quad \partial^\kappa = \partial_1^{\kappa_1} \dots \partial_d^{\kappa_d}$$

- The power series defining \star_M and \star_V generally diverge for functions in S
- The Moyal product can be continuously extended from a suitable subspace to S
- For the Wick-Voros product such an extension is impossible

Twisted (Moyal) tensor product

$$(f \otimes_{\theta} g)(x, y) = f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_{\mu} \theta^{\mu\nu} \overrightarrow{\partial}_{\nu}\right) g(y)$$

$$(f \star_{\theta} g)(x) = (f \otimes_{\theta} g)(x, x)$$

$$(f \otimes_{\theta} g) \otimes_{\theta} h = f \otimes_{\theta} (g \otimes_{\theta} h)$$

$$(f \otimes_{\theta} g)(x_1, \dots, x_m; y_1, \dots, y_n) = \prod_{a=1}^m \prod_{b=1}^n e^{\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x_a^{\mu}} \frac{\partial}{\partial y_b^{\nu}}} f(x_1, \dots, x_m) g(y_1, \dots, y_n)$$

$$(f \widehat{\otimes}_{\theta} g)(p, q) = \underbrace{\exp\left(\frac{i}{2} \theta^{\mu\nu} p_{\mu} q_{\nu}\right)}_{\uparrow} (\widehat{f} \otimes \widehat{g})(p, q), \quad f \in S(\mathbb{R}^d), \quad g \in S(\mathbb{R}^d)$$

multiplier of $S(\mathbb{R}^{2d})$

Noncommutative deformations of quantum field theories

Let ϕ be a scalar field on commutative Minkowski space and

$$\langle \Psi_0, \phi(f_1) \cdots \phi(f_n) \Psi_0 \rangle = (W^{(n)}, f_1 \otimes \cdots \otimes f_n), \quad W^{(n)} \in S'(\mathbb{R}^{4n})$$

$$(W_\theta^{(n)}, f_1 \otimes \cdots \otimes f_n) \stackrel{\text{def}}{=} (W^{(n)}, f_1 \otimes_\theta \cdots \otimes_\theta f_n)$$

$$\Phi_n(f) = \int dx_1 \dots dx_n \phi(x_1) \cdots \phi(x_n) f(x_1, \dots, x_n) \Psi_0, \quad f \in S(\mathbb{R}^{4n})$$

We set

$$\phi_\theta(g) \Psi_0 = \phi(g) \Psi_0, \quad \phi_\theta(g) \Phi_n(f) = \Phi_{n+1}(g \otimes_\theta f), \quad n \geq 1$$

Then

$$\langle \Psi_0, \phi(x_1) \cdots \phi(x_n) \Psi_0 \rangle = W_\theta^{(n)}(x_1, \dots, x_n)$$

and the vacuum state Ψ_0 is cyclic for every field ϕ_θ

Th. 1. [Phys. Rev. D 77 (2008) 125013] Let ϕ be a Hermitian scalar field satisfying the usual Wightman axioms. Then every deformed field ϕ_θ is well-defined as an operator-valued tempered distribution with the same domain in the Hilbert space. Moreover,

$$\phi_\theta(g)^* = \phi_\theta(\bar{g}) \quad \text{for all } g \in S(\mathbb{R}^4)$$

and

$$\sum_{m,n=1}^N (W_\star^{(m+n)}, f_m^\dagger \otimes f_n) \geq 0, \quad \text{for all } f_m \in S(\mathbb{R}^{4m}), f_n \in S(\mathbb{R}^{4n})$$

(with $f^\dagger(x_1, \dots, x_n) \stackrel{\text{def}}{=} \overline{f(x_n, \dots, x_1)}$).

Deformation of a free field

$$a_\theta(p) = e^{(i/2)p\theta P} a(p), \quad a_\theta^*(p) = e^{-(i/2)p\theta P} a^*(p),$$

where P is the energy-momentum operator

$$a_\theta(p)a_\theta(p') = e^{-ip\theta p'} a_\theta(p')a_\theta(p)$$

$$a_\theta^*(p)a_\theta^*(p') = e^{-ip\theta p'} a_\theta^*(p')a_\theta^*(p)$$

$$a_\theta(p)a_\theta^*(p') = e^{ip\theta p'} a_\theta^*(p')a_\theta(p) + 2\omega_{\mathbf{p}}\delta(\mathbf{p} - \mathbf{p}')$$

Transformation law of the deformed fields under the Poincaré group

$$U(y, \Lambda)\phi_\theta(x)U^{-1}(y, \Lambda) = \phi_{\Lambda\theta\Lambda^T}(\Lambda x + y), \quad \Lambda \in L_+^\uparrow$$

Violation of microcausality

If $\Phi = \varphi^{(-)}(h_1)\varphi^{(-)}(h_2)\Psi_0$, $h_{1,2} \in S(\mathbb{R}^4)$, then

$$M_\Phi(x, y) = \langle \Psi_0, [\varphi_\theta(x), \varphi_\theta(y)] \Phi \rangle \neq 0 \text{ for } (x - y)^2 < 0.$$

Moreover, $\text{supp } \hat{M}_\Phi \subset \bar{V}_+ \times \bar{V}_+ \implies \text{supp } M_\phi = \mathbb{R}^8$

Localization in wedges

(Grosse, Lechner, JHEP 0809:131,
2008)

$$\theta = \begin{pmatrix} 0 & \theta_e & 0 & 0 \\ -\theta_e & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_m \\ 0 & 0 & -\theta_m & 0 \end{pmatrix} \longrightarrow W_1 = \{x \in \mathbb{R}^4 : x^1 > |x^0|\}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \Lambda\theta\Lambda^T & \longleftrightarrow & \Lambda W_1 \end{array}$$

If the sets $x + W_\theta$ and $y + W_{\theta'}$ are spacelike separated, then

$$[\phi_\theta(x), \phi_{\theta'}(y)] = 0$$

A causal wedge in place of the light cone

$$\begin{aligned} \mathcal{O}(x) \stackrel{\text{def}}{=} : \varphi \star \varphi : (x) &= \lim_{x_1, x_2 \rightarrow x} : \varphi(x_1) \varphi(x_2) : \\ + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \dots \theta^{\mu_n \nu_n} &\lim_{x_1, x_2 \rightarrow x} : \partial_{\mu_1} \dots \partial_{\mu_n} \varphi(x_1) \partial_{\nu_1} \dots \partial_{\nu_n} \varphi(x_2) : \end{aligned}$$

In the case of space-space noncommutativity ($\theta^{23} = -\theta^{32} \neq 0$ and $\theta^{\mu\nu} = 0$ for $\mu, \nu \neq 2, 3$) the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ vanishes in the wedge $|x^0 - y^0| < |x^1 - y^1|$, but

$$\langle 0 | [\mathcal{O}(x), \partial_0 \mathcal{O}(y)]_- | p_1, p_2 \rangle |_{x^0=y^1} \neq 0 \quad (\text{Greenberg, Phys. Rev. D, 2006})$$

Th. 2. Let $\mathcal{O}(x)$ be defined via the Moyal \star -product with $\theta^{23} = -\theta^{32} \neq 0$ and the other elements of the θ -matrix equal to zero. Then $[\mathcal{O}(x), \mathcal{O}(y)] \neq 0$ everywhere outside the wedge $|x^0 - y^0| < |x^1 - y^1|$ and the star commutator $[\mathcal{O}(x), \mathcal{O}(y)]_\star \stackrel{\text{def}}{=} \mathcal{O}(x) \star \mathcal{O}(y) - \mathcal{O}(y) \star \mathcal{O}(x)$ does not vanish for all x, y .

The case of Wick-Voros product

Let $\mathcal{O}(x) =: \varphi \star_V \varphi : (x)$, $\theta^{23} = -\theta^{32} = \theta \neq 0$, and $\Phi = \varphi^{(-)}(h_1)\varphi^{(-)}(h_2)\Psi_0$.

Then

$$\begin{aligned} \langle \Psi_0, [\mathcal{O}(x), \mathcal{O}(y)] \Phi \rangle &= 8 \int \frac{dk dp_1 dp_2}{(2\pi)^{3(d-1)}} \epsilon(k_0) \delta(k^2 - m^2) e^{\frac{\theta}{2} \mathbf{k} \cdot (\mathbf{p}_2 - \mathbf{p}_1)} \\ &\quad \times e^{-ik \cdot (x-y) - ip_1 \cdot x - ip_2 \cdot y} \prod_{i=1}^2 \vartheta(p_{i0}) \delta(p_i^2 - m^2) \cos\left(\frac{1}{2} k \theta p_i\right) \hat{h}(p_i), \end{aligned}$$

where $\mathbf{k} = (k_2, k_3)$, $k \theta p = k_\mu \theta^{\mu\nu} p_\nu$.

Because of the factor $e^{\frac{\theta}{2} \mathbf{k} \cdot (\mathbf{p}_2 - \mathbf{p}_1)}$ this expression is not a tempered distribution can be defined only on analytic test functions

Conditions for convergence of the star products

$$(1 + |x|)^N |\partial^\kappa f(x)| < C_N B^{|\kappa|} (\kappa!)^{1/2},$$

$$B < \frac{1}{\sqrt{|\theta|}}, \quad |\theta| = \sum |\theta^{\mu\nu}|$$

Def. 1. A smooth function f on \mathbb{R}^d belongs to $\mathcal{S}^{1/2}(\mathbb{R}^d)$ if for each $B > 0$ and for any integer N , there exists a constant $C_{B,N}$ such that

$$(1 + |x|)^N |\partial^\kappa f(x)| < C_{B,N} B^{|\kappa|} (\kappa!)^{1/2}$$

We endow $\mathcal{S}^{1/2}$ with the topology determined by the set of norms

$$\|f\|_{B,N} = \sup_{x,\kappa} (1 + |x|)^N \frac{|\partial^\kappa f(x)|}{B^{|\kappa|} (\kappa!)^{1/2}}$$

Under this topology $\mathcal{S}^{1/2}$ is a nuclear Fréchet space

Test function spaces for NC QFT

Th. 3. [Theor. Math. Phys. 153 (2007); J. Phys. A 40 (2007)]

The space $\mathcal{S}^{1/2}(\mathbb{R}^d)$ is a topological algebra under the Moyal \star -product as well as under the Wick-Voros \star -product. If $f, g \in \mathcal{S}^{1/2}(\mathbb{R}^d)$, then the series representing these products converge absolutely in this space. Moreover these products depend continuously on the noncommutativity parameter θ .

$\mathcal{S}^{1/2}$ is largest of the subspaces of the Schwartz space that have such properties

We also use the space $S_{1/2,A}^{1/2,B}(\mathbb{R}^d)$ of all smooth functions on \mathbb{R}^d with the property that

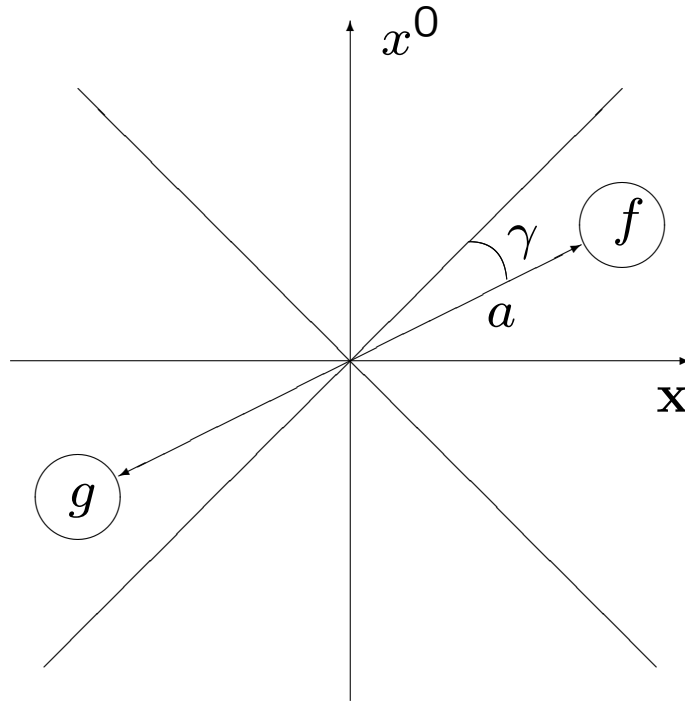
$$\|f\|_{A,B} = \sup_{\kappa,x} e^{|x/A|^2} \frac{|\partial^\kappa f(x)|}{B|\kappa|_{\kappa}^{\kappa/2}} < \infty$$

This space is nontrivial if $AB > 2$

Causal commutator of averaged observables

$$\mathcal{O}(f_a) = \int dx \mathcal{O}(x) f(x - a), \quad f \in S_{1/2,A}^{1/2,B}$$

$$\langle 0 | [\mathcal{O}(f_a), \mathcal{O}(g_{-a})] | \Phi \rangle, \quad |\Phi\rangle = \varphi^-(h_1) \varphi^-(h_2) |\Psi_0\rangle$$



$$\gamma = \inf_{\xi^2 \geq 0} \left| \xi - \frac{a}{|a|} \right|$$

Th. 4. Let ϕ be a free scalar field and let $\mathcal{O}(x) = : \phi \star \phi : (x)$, where the \star -product is determined by an arbitrary real antisymmetric matrix $\theta^{\mu\nu}$. Suppose that test functions f and g belong to $S_{1/2,A}^{1/2,B}$, where $A > 0$ and $0 < B < 1/\sqrt{|\theta|}$. Then

$$|\langle 0 | [\mathcal{O}(f_a), \mathcal{O}(g_{-a})] | \Phi \rangle| \leq C_{\Phi, A'} \|f\|_{A,B} \|g\|_{A,B} e^{-2|\gamma a/A'|^2}$$

for each $A' > A$.

Because of the conditions $B < 1/\sqrt{|\theta|}$ and $AB > 2$, the best result is at $A \sim 2\sqrt{|\theta|}$ and demonstrates a decrease

$$\sim \exp\left(-\frac{|\gamma a|^2}{2|\theta|}\right)$$

The same result holds in the case of Wick-Voros product

Def. 2. Let V be a cone in $\subset \mathbb{R}^d$. A smooth function on \mathbb{R}^d belongs to the space $S^{1/2,B}(V)$, if it satisfies the condition

$$\sup_{x \in V} (1 + |x|)^N |\partial^\kappa f(x)| < C_N B^{|\kappa|} (\kappa!)^{1/2},$$

$$\mathbb{V} = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : (x - y)^2 \geq 0\}.$$

θ -locality condition: For any fields φ, ψ and for any states Φ, Ψ in a common invariant domain $D \subset H$, either

$$\langle \Phi, [\phi(x), \psi(x')]_- \Psi \rangle$$

or

$$\langle \Phi, [\phi(x), \psi(x')]_+ \Psi \rangle$$

can be continuously extended to the space $S^{1/2,B}(\mathbb{V})$, where $B_{\varphi,\psi,\Phi,\Psi} \sim 1/\sqrt{|\theta|}$.

Conclusions

- The noncommutative deformation of QFT by twisting tensor products leads to the lack of microcausality, though preserves (in the Moyal case) certain relative localization properties
- The space $\mathcal{S}^{1/2}$ is universal for a nonperturbative (in particular for a Wightman-type axiomatic) formulation of NC QFT
- This space, being a maximal topological star product algebra with absolute convergence, is completely adequate to the concept of strict deformation quantization
- The θ -locality condition, which means heuristically that the commutators of observables behave at large spacelike separation like

$$\exp(-|x - y|^2/\theta)$$

can possibly be used for formulating causality in NC QFT